Positive and Negative Integers

Grades 5 - 8

Student Understanding of Magnitudes and Basic Operations

Pedagogy

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Student Understanding of Magnitudes and Basic Operations - Pedagogy

After several visits to middle schools in the last couple months, my interest in the pedagogy of positive and negative integers has dramatically increased. From a curriculum sequencing perspective, positive and negative numbers are introduced at the end of 5th grade in select elementary schools and fully during sixth grade. The mathematical content of negative and positive integers is embryonic, so background information does not hinder student understanding as it often does with other math content areas – meaning a vast number of dependent numeracy skills is not vertically aligned or needed to comprehend the rudiments of positive and negative numbers. Furthermore, The Khan Academy and many YouTube videos offer the typical classroom teacher rich, in depth course content and invaluable pedagogical advice; hence, it seems more than a little surprising that so many students seem to struggle with this important number sense concept.

After much reflection, I believe one of the problematic pedagogical issues in student understanding of negative and positive integer magnitudes and operations may be due to the possibility that a number line was not used in the initial instructional lessons to the extent that was necessary. Therefore, I created a basic template with positive and negative numbers that ranges from negative twelve (-12) to positive twelve (+12) on 11 x 17 inch paper so the student aid is sufficiently large for effective classroom use. There are five (5) templates on each page so the teacher may have a classroom set with only six (6) Xeroxed copies. The positive and negative integer number line template is included at the end of this document.

This short pedagogy paper is intended to supplement a math teacher's lesson preparation and thinking with regard to teaching positive and negative integers' magnitudes and their use in basic math operations (i.e. addition, subtraction, multiplication and division). This instruction is highly dependent upon the daily inclusion of incorporating a student aid positive and negative integer number line until the concepts are mastered. The use of a positive/negative integer number line template is essential for understanding relative magnitudes of negative numbers as well as their inclusion in the physical meaning of addition and subtraction. This number line is easy and quick to prepare each day, so students may use the number lines until a true understanding of positive and negative integer relationships is thoroughly developed. Furthermore, the implementation of spaced repetition instruction into the learning process is also essential to ensure students are exposed to threshold levels of repetition to secure mastery of the content. A free downloadable white paper and two blogs with more detailed information are available for the interested educator at the website address in the footer of this document.

Relative Magnitudes of Positive and Negative Integers - Comparing using < , > , = Signs

If students do not understand relative magnitudes of positive and negative numbers, a majority of related mathematical content presented in future math lessons will not be founded on mathematical bedrock. Subsequently, there is one general rule that students should completely understand when considering the magnitude or the relative size of integers regardless of its fixed location/position on the number line.

General Rule of Magnitude for All Integers: "***The integer to the right is a larger integer.***"

As students learned in their elementary math classes, the integer that is farthest to the right is the largest in value. As expected this situation does not change when considering the relative magnitudes of positive or negative integers. In fact, there are only six unique situations.

There are six (6) general cases to consider with *comparing* positive and negative integers. Those six (6) unique cases are the following: (Case 1: Positive – Positive), (Case 2: Positive – Zero), (Case 3: Negative – Negative), (Case 4: Negative – Zero), (Case 5: Positive – Negative) and (Case 6: Equal Integers – Positive or Negative). Specific examples of each case with a numerical situation is shown below.

Case 1: If given two integers, 3 and 5. Five (5) is farther to the right on the number line; therefore, it is larger in value than three (3). Hence, 5 > 3 or 3 < 5.

Case 2: If given two integers, 0 and 8. Eight (8) is farther to the right on the number line; therefore, it is larger in value than zero (0). Hence, 8 > 0 or 0 < 8.

Case 3: If given two integers, 7 and - 3. Seven (7) is farther to the right on the number line; therefore, it is larger in value than negative three (- 3). Hence, 7 > - 3 or - 3 < 7.

Case 4: If given two integers, - 6 and 0. Zero (0) is farther to the right on the number line; therefore, it is larger in value than negative six (- 6). Hence, 0 > - 6 or - 6 < 0.

Case 5: If given two integers, - 4 and - 2. Negative (- 2) is farther to the right on the number line; therefore, it is larger in value than negative four (- 4). Hence, - 2 > - 4 or - 4 < - 2.

Case 6: If given two integers, - 9 and - 9. Negative nine (- 9) is in the exact same location as negative nine (- 9). Hence, - 9 = - 9.

Second semester 5th graders and middle school students must completely understand the relative magnitudes of positive and negative numbers to correctly add and subtract integers. It is highly recommended that a teacher devote a couple class days of instruction in this content area prior to transitioning to addition and subtraction of positive and negative integers. The lessons on magnitude and other concepts are dramatically heightened with short mini-lessons (see Spaced Repetition Instruction methodology as referenced on website address in the footer) to ensure students understand the relative size of positive and negative integers.

Addition of Positive and Negative Integers

Addition of two integers regardless if the integers are negative or positive is a fairly straight forward procedure. If the two integers are both positive (i.e. + 2, + 5), then the addition problem is identical to the addition problems the student has successfully engaged since kindergarten. The two integers represent the total number of combined spaces on the number line when summed. As expected, the same physical meaning is true when one integer is positive and one integer is negative. For example, if the addition problem is the following equation: 5 + (-3), then the student can sum the integers by using the location of either integer as a starting point, and move in the direction of positive or negative value of the other integer. For instance, on 5 + (-3), students may begin on the number line at the number 5 (or count 5 spaces from zero to locate positive five (+ 5), but not necessary for older students) and move 3 spaces to the left, since it is a negative 3 to obtain the sum of +2. Or, start at negative three (-3) on the number line, and move 5 spaces to the right since five (5) is positive, and the same sum of +2 is obtained.

It is highly recommended that students use a number line when working problems until they are highly proficient at the process and can mentally map a number line with positive and negative integers. Implementing a spaced repetition instructional system using addition of positive and negative numbers is an effective means to ensure student mastery. Prior to the start of the core math lesson, the teacher should provide 3 to 5 addition problems using both negative and positive operations. The problems should be quickly reviewed and checked by the classroom teacher. In 4 to 6 consecutive days of consistent instruction, students have mastered the positive and negative integer addition.

The spaced repetition should continue as students are introduced to subtraction of negative and positive numbers. Although, after the subtraction process has been introduced and students have demonstrated a basic understanding, the teacher should include one or two subtraction problems in conjunction with the addition problems. Students should be presented with ample opportunities to become proficient. As they complete both addition and subtraction problems over multiple days of the mini-lesson/spaced repetition, students begin to separate the operational processes of addition and subtraction and ingrain their unique meaning until they are competent completing each problem without the use or need of a number line visual aid. Finally, it is important to note that when students begin to learn subtraction of negative and positive numbers, many students will confuse the previously learned addition process and make errors; however, with continued daily practice with addition as they learn subtraction, these errors significantly diminish.

Subtraction of Positive and Negative Integers

Most elementary students discover the following truism by first grade: Subtraction is much more challenging than addition. Unfortunately, this axiom does not change when negative integers are added into the fray. There are two main reasons students generally struggle more on subtraction than addition. First, the commutative property of addition eliminates memorizing many of the standard one-digit addition facts since 5 + 3 = 3 + 5. Simply put, it matters not which of the two integers a student selects to begin the addition process to compute the sum of any two integers. Unfortunately, there is not a commutative property for subtraction. Second, when subtracting two integers, students often do not realize that they are computing a ‘difference’ between the integers. In fact, students are calculating the total number of ‘spaces’ on the number line between the two integers. This aspect of the subtraction of two positive integers is true in all subtraction processes. The student is always computing the number of spaces or distance between two integers regardless if the integers are positive or negative. Consequently, **the first step** in computing differences between positive and negative integers is computing the total number of spaces between them on the number line. **The second step** is determining if the difference is negative or positive. For example, when computing the difference between 7 – 4, there are 3 spaces between the two integers. Consequently, the first step of the subtraction process yields that the difference is 3. The second step depends upon the relative magnitude of the two integers. Since 4 is smaller than 7, the difference must be positive. The second step in the subtraction process was an oblivious step to elementary students since they always subtracted a smaller integer from a larger one.

However, if one of the integers is a positive integer and the other is a negative integer, the process is exactly the same, but students must evaluate if the value of the difference is positive or negative. For instance, in the subtraction problem, 3 – (-4), there are 7 total spaces between positive three (+3) and negative four (- 4). Therefore, the difference is 7. But, when working with positive and negative integers, there is a second step. Since a smaller integer (-4) is subtracted from a larger integer, the difference is positive. If the subtraction problem were reversed, (-4) – (+3), then the total number of spaces or difference remains 7 spaces between the two integers, but the difference is a negative seven (-7) since we are subtracting a larger number from a smaller number.

The last case to consider is when students are subtracting two negative integers from one another. For example, give the subtraction problem, (-3) – (-8), there are 5 total spaces between the two negative integers; hence, the difference is five (5). In step 2, since negative eight (-8) is smaller than negative three (-3), the student is subtracting a smaller number from a larger number and the difference is a positive five (+5).

If students are not provided instruction on the physical meaning of subtraction that matches their understanding from his or her primary grades, students think subtraction possesses a different physical meaning than they initially learned. In sixth grade, students often are taught to memorize that when subtracting a negative number from a positive number, that they should automatically add without realizing that they are actually computing the total number of spaces between the two integers. The subtraction process yields the correct answer, but it does not result in mathematical understanding of the process. This lack of foundational understanding often hinders the successful systematic construction of dependent mathematics in subsequent middle and high school math courses.

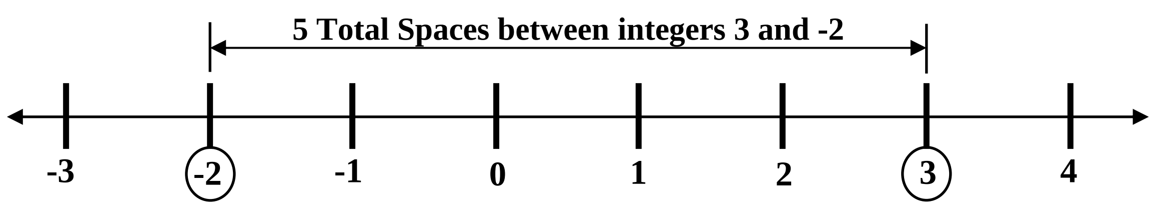


Figure A

Figure A above illustrates a subtraction process for the problem/equation: 3 – (-2) = 5. In this situation, the student completely understands computing differences with positive and negative integers. The two integers, positive three (+ 3) and negative two (- 2), are both circled on the whole number line. There are five total spaces between the two integers. Since positive three (+3) is larger than negative two (- 2), the difference is a positive five or +5.

However, if the subtraction problem shown in Figure A were reversed to the following equation: -2 – (+3), then there are still 5 total spaces between the integers, but since positive three (+ 3) is larger than negative two (- 2), the difference is a negative five or - 5.

In summary, there are always **two steps** when computing differences that promote complete student understanding of the subtraction process when using positive and negative numbers.

First, compute the total number of spaces (the difference) between the two integers on a number line either mentally or initially, with the aid of a negative/positive number line. This meaning is identical to the subtraction process that was learned in the primary grades during the students' elementary years.

Second, the student analyzes the subtraction problem to determine if the difference computed in step 1 is either a positive or negative value. If a smaller integer is subtracted from a larger integer, the difference value is positive. If a larger integer is subtracted from a smaller integer, the difference value is negative.

Multiplication and Division of Positive and Negative Integers

Multiplication and Division of positive and negative numbers is extremely straightforward and much easier to understand than the subtraction process. The student only needs to know the sign conventions for multiplication and division. If the two factors are positive, then the product (or quotient if dividend and divisor are of opposite signs) is positive. If one of the factors is negative and the other positive, then the product (or quotient) is negative. Finally, if both factors are negative, then the product (or quotient) is positive. Students usually memorize these situations, and if they are familiar with their multiplication and division math facts, they are functionally able to compute products, factors, and quotients correctly.

However, for the intention of a more thorough analysis, the mathematical reason these sign conventions exist can be illustrated based the following proof using the distributive property. If mathematics is to be consistent, then the distributive property of multiplication must remain valid when computing negative and positive products.

For example, if the following distributive property is true and must remain valid in all operations, then a positive integer multiplied by a negative integer **must** equal a negative product.

Proof that a negative integer multiplied by a positive integer equals a negative product

3(0) = 0. Simply put, a whole number multiplied by zero (0) must equal zero (0). If zero is rewritten as adding a positive integer to a negative integer of the opposite value, then (6 + (-6)) = 0.

3(6 + (-6)) = 0 ⟹ expanding via the distributive property yields: 3(6) + 3(-6) = 0

In order to be mathematically true, 18 + 3(-6) = 0 Hence 3(-6) must be a -18. ***Therefore, a negative integer multiplied by a positive integer must always equal a negative product.***

Proof that a negative integer multiplied by a negative integer equals a positive product

This same analysis can be developed to prove that a negative integer multiplied by another negative integer must be a positive product. Again, using the distributive property of multiplication to prove that premise is illustrated below.

-3(6 + (-6)) = 0 ⟹ expanding via the distributive property yields: -3(6) + (-3)(-6) = 0

In order to be mathematically valid, -18 + (-3)(-6) = 0 Hence, (-3)(-6) must be a positive 18. ***Therefore, a negative integer multiplied by a negative integer must always equal a positive product***.

As expected, the same sign convention rules are valid for either multiplication or division computations. The multiplication and division sign conventions rules are summarized below:

<u>Multiplication and Division Sign Conventions</u>

1.) **[x or ÷]: <u>Positive</u> (+) by <u>Positive</u> (+) = <u>Positive</u> Product/Quotient**

2.) **[x or ÷]: <u>Negative</u> (-) by <u>Positive</u> (+) = <u>Negative</u> Product/Quotient**

3.) **[x or ÷]: <u>Negative</u> (-) by <u>Negative</u> (-) = <u>Positive</u> Product/Quotient**

Final Comments Concerning Positive and Negative Integers

It is imperative that students readily understand positive and negative integers and their physical meaning in both magnitudes and operations. In order to accomplish this objective, students require consistent, quality instruction on set processes. The use of a number line student aid that possesses positive and negative numbers significantly assists students to visually understand the mathematics and the corresponding physical meaning. Students must possess a visualization of a 'mental positive/negative number line' when working integer addition, subtraction, multiplication or division operations regardless of the integer sign. Students should not rote memorize but fully comprehend that the same mathematical rules that were valid during their elementary school numeracy work with only positive integers are identical to the same mathematical rules that govern the operations of positive and negative integers.

The end result of an in-depth student understanding of negative and positive integer work is significant and lasting. The process ensures that the mathematics instruction presented in later grades builds upon a correctly constructed foundation of mathematical conceptual and physical meaning.

With the use of 'spaced repetition instruction,' students are systematically exposed to the threshold number of repetitions required to master mathematics content. A white paper expounding on the basics and benefits of 'spaced repetition' can be downloaded for free at the website address located in the footer. This instructional method is a highly effective and efficient pedagogical process that ensures both mastery and retention of prior content. Finally, a recommended sequence of instruction is summarized below to assist teachers in the pedagogical planning and preparation process.

<u>Positive and Negative Integer – Recommended Sequencing - Pedagogy</u>

1.) ***Teacher secures all students undivided attention prior to beginning daily lesson*.**

2.) **Using the positive/negative number line, the teacher should model (direct teach), provide guided instruction and independent practice on <u>comparing two integers</u> using < , > , and = signs. Continue until students can successfully engage problems without the need of the number line. Check Understanding. Duration: 2 to 3 days of instruction, adjust as needed.**

3.) **Using the positive/negative number line, the teacher should model (direct teach), provide guided instruction and independent practice on the <u>addition of two integers</u>. Continue until students can successfully engage problems without the need to use the number line. Again, check student understanding! Duration: 3 to 5 days of instruction, adjust as needed.**

<u>Positive and Negative Integer – Recommended Sequencing – Pedagogy</u> *CONTINUED*

4.) **The classroom teacher should consider providing three (3) to six (6) comparing integer problems *daily* at the onset of each addition of integer's core lessons to ensure mastery of previously taught content – Spaced Repetition Instruction. Then, gradually interject the addition integer problems and phase out daily instruction on the comparison of integer problems to only one (1) per day. Duration: Mini-Lesson, every day, adjust content as needed.**

5.) **Using the positive/negative number line, the teacher should model (direct teach), provide guided instruction and independent practice on the <u>subtraction of two integers</u>. Continue until students can successfully engage problems without the need to use the number line. Check student understanding. Duration: 5 to 7 days of instruction, adjust as needed.**

6.) **The classroom teacher continues providing three (3) to six (6) addition of integer problems at the onset of subtraction of integer core lessons to ensure mastery of previously taught content – Spaced Repetition Instruction. Gradually interject both the addition and subtraction integer problems. Duration: Mini-Lesson, every day, adjust content as needed.**

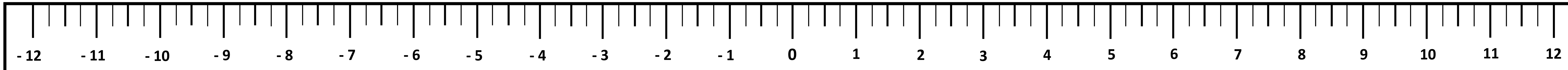
7.) **Using the positive/negative number line (model as multiplication as repetitive adding in either the positive or negative number line direction), the teacher should model (direct teach), provide guided instruction and independent practice on the <u>multiplication of two integers</u>. Continue until students can successfully engage problems without the need to use the number line. Duration: 4 to 6 days of instruction, adjust as needed.**

8.) **The classroom teacher continues providing five (5) to nine (9) addition and subtraction of integer problems prior to the core lessons to ensure mastery of previously taught content – Spaced Repetition Instruction. Gradually reduce addition problems and subtraction integer problems and interject multiplication problems of two integers. Further challenge students with a daily diet of one (1) to three (3) distribution property of multiplication application problems. For example: 5(x -2) and -3(-4 + y). Duration: Spaced Repetition/Daily warm-up or Mini-Lesson, every instructional day, adjust content as needed.**

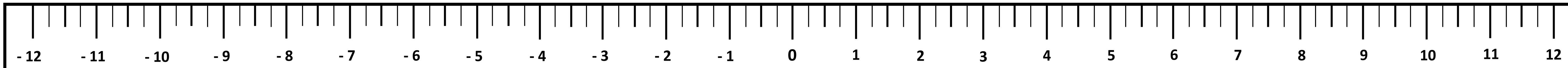
9.) **The teacher should model (direct teach), provide guided instruction and independent practice on the <u>division of two integers</u>. Continue until students can successfully engage problems. Teacher may use the number line. For example, select the integer, negative 8 (-8) divided by four (4). Quotient is negative 2 (-2). Consequently, there are four (4) equal groups of negative two (-2). Duration: 3 to 4 days of instruction, adjust as needed.**

10.) **The classroom teacher should continue with the daily spaced repetition instruction/mini-lessons using one (1) comparison of integers, one (1) addition problem of integers, one (1) subtraction problem of integers, one (1) multiplication problem of integers, two (2) distributive property of multiplication problems, and two (2) to three (3) division of integer problems until students demonstrate proficiency of all content. These daily spaced repetition/mini-lesson formative assessments will indicate the students' preparedness to successfully be engaged in a final summative assessed on all negative and positive integer work. The teacher should assist any and all students struggling during the daily spaced repetition/mini-lesson via active monitoring of students. *<u>Always check student understanding before moving on!</u>* Duration: 2 to 3 days of instruction, adjust as needed.**

POSITIVE AND NEGATIVE NUMBERS – NUMBER LINE - STUDENT AID (NOTE: On 11” x 17” Paper)



Positive and Negative Integer Number Line



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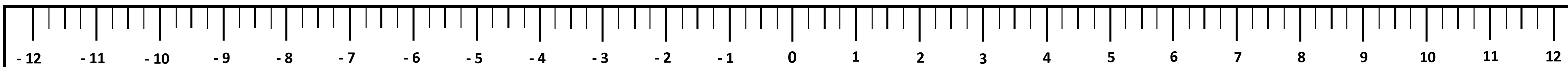

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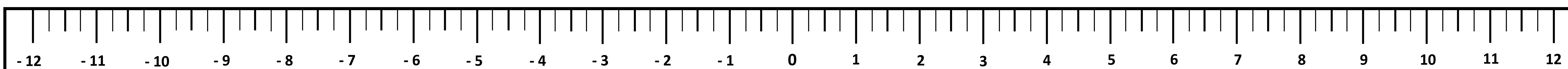

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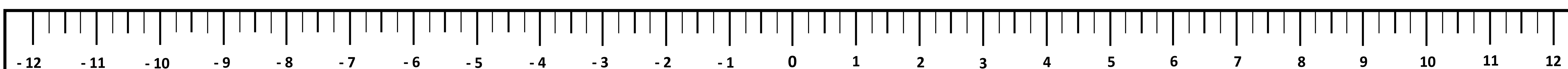

Positive and Negative Integer Number Line